Initial Value Problems for ODE's

(IVP)

We want numerical methods for solving ODE's of the form:

$$\left.\begin{array}{l} \dfrac{dy}{dt} = f(t, y) \\ \text{for } a \le t \le b \quad \text{with } y(a) = \alpha \end{array}\right\} \text{——— } (*)$$

i.e., we want numerical methods to approximate the solution $y(t)$ of $(*)$.

More generally:

n^{th} order ODE

$$y^{(n)} = f(t, y', y'', \dots, y^{(n-1)})$$

for $t \in [a, b]$ with $y(a) = \alpha_1$, $y'(a) = \alpha_2$, $\dots$, $y^{(n-1)}(a) = \alpha_n$

and

System of ODEs

$$\frac{dy_i}{dt} = f_i(t, y_1, y_2, \dots, y_n) \; ; \quad i = 1, \dots, n$$

with $t \in [a, b]$, $y_i(a) = \alpha_i$, $i = 1, \dots, n$

One basic idea we will follow is to approximate the sol'n at certain pts $\tilde{y}(t_1), \ldots, \tilde{y}(t_m)$

and use interpolation to get the function $\tilde{y}(t) \approx y(t)$.

Before we dive in, we need some background theory:

Def'n: $f(t,y)$ is Lipschitz in y on $D \subset \mathbb{R}^2$, with constant L, if

$$|f(t,y_1) - f(t,y_2)| \leq L|y_1 - y_2|$$

$\forall (t,y_1), (t,y_2) \in D$.

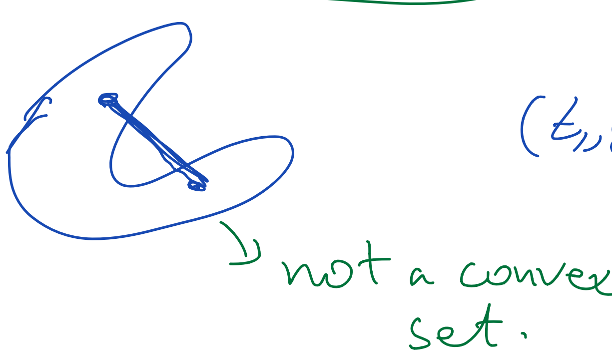
Example: $f(t,y) = t|y|$; $D = [1,2]\times[-3,4]$ is Lipschitz bec.

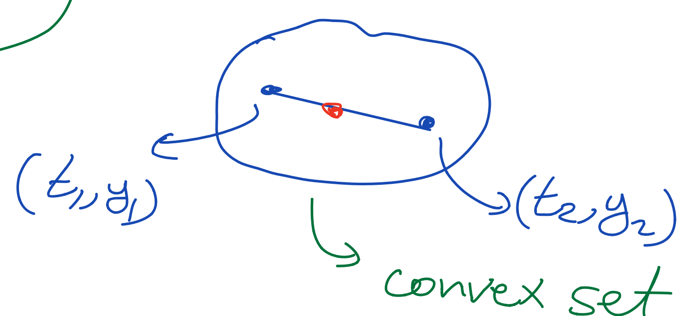
$$|f(t,y_1) - f(t,y_2)| = |t|\,\big||y_1| - |y_2|\big|$$
$$\leq \underbrace{2}_{\text{Lipschitz const.}}|y_1 - y_2| \quad \blacksquare$$

Def'n: $D \subset \mathbb{R}^2$ is convex if $\forall (t_1,y_1)$ & $(t_2,y_2) \in D$ we have

$$(1-\lambda)(t_1,y_1) + \lambda(t_2,y_2) \in D$$

$\forall \lambda \in [0,1]$.





Theorem: If $f: D \subset \mathbb{R}^2 \to \mathbb{R}$ satisfies
convex

$$\left| \frac{\partial f}{\partial y}(t,y) \right| \leq L \quad \forall (t,y) \in D$$

then f is Lipschitz on D with constant L.

Sufficient cond. for a fct to be Lipschitz.

Theorem: Let $D = [a,b] \times \mathbb{R}$ and let f be continuous on D. If f is Lipschitz in the variable y on D then the IVP

$$\begin{cases} y'(t) = f(t,y), & t \in [a,b] \\ y(a) = \alpha \end{cases}$$

has a unique sol'n $y(t)$ for $t \in [a,b]$

Example: Use the theorem to show that $\exists$ a unique solution to the IVP

$$\begin{cases} y'(t) = 1 + t\sin(ty), & 0 \le t \le 2 \\ y(0) = 0 \end{cases}$$

Sol'n: The function $f(t,y) = 1 + t\sin(ty)$ is continuous on $[0,2] \times \mathbb{R}$ & has $\left|\frac{\partial f}{\partial y}\right| = \left|t^2\cos(ty)\right| \le 4$ on $t \in [0,2]$

$\Rightarrow$ f is Lipschitz with const 4

so by the theorem, the IVP has a unique sol'n.

Well Posedness:

The IVP : $\begin{cases} \dfrac{dy}{dt} = f(y,t), \ t \in [a,b] \\ y(a) = \alpha \end{cases}$

is well-posed if

(1) $\exists$ a unique sol'n $y(t)$

(2) The perturbed problem

$$\begin{cases} \dfrac{dz}{dt} = f(z,t) + \delta(t), \ t \in [a,b] \\ z(a) = \alpha + \delta_0 \end{cases}$$

also has a unique solution

$z(t)$ with $|z(t) - y(t)| < k\varepsilon$

$\forall$ continuous $\delta(t)$ with $|\delta(t)| < \varepsilon$
& $\delta_0 < \varepsilon$
$\in [a,b]$

Why well-posedness?

- Modeling errors
- Round-off errors
- measurement errors ...

Theorem: If f is cont. on D where $D = [a,b] \times \mathbb{R}$, and if f is Lipschitz in y on D, the

IVP: $\begin{cases} \frac{dy}{dt} = f(t,y), & t \in [a,b] \\ y(a) = \alpha \end{cases}$

is well posed.

Euler's Method:

IVP: $\begin{cases} \frac{dy}{dt} = f(t,y), & t \in [a,b] \\ y(a) = \alpha \end{cases}$

- Pick $N+1$ equispaced points

  $t_0, \dots, t_N$ ⟵ mesh points

with

$$t_i = a + (i-1)\underset{\uparrow \text{ step size}}{h}, \quad \text{for } h = \frac{b-a}{N}$$

and note that

$$\underbrace{y(t_{i+1})}_{\downarrow \; w_{i+1}} = \underbrace{y(t_i)}_{\downarrow \; w_i} + \underbrace{h y'(t_i)}_{\downarrow \; h f(t_i, w_i)} + \underbrace{\frac{h^2}{2} y''(\overset{\in [t_i, t_{i+1}]}{\xi_i})}_{\downarrow \; \text{approx with } 0}$$

$$\Longrightarrow \boxed{\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + h f(t_i, w_i) \\ \quad \text{for } i = 0, \dots, N-1 \end{cases}}$$

↑ Difference Eq'n.

Now, we have approx. values of y at $t_0, \dots, t_N$, so we can interpolate to find an approx. of y.

Example: $\begin{cases} y' = y - t^2 + 1 & t \in [0,2] \\ y(0) = 0.5 \end{cases}$

Pick $h = 1$ & use Euler's method:

$\Rightarrow t_0 = 0, \quad t_1 = 1, \quad t_2 = 2$

$w_0 = 0.5$

$$w_1 = w_0 + h f(t_0, w_0)$$
$$= 0.5 + 1 \times (0.5 - 0^2 + 1) = 2$$

$$w_2 = w_1 + h f(t_1, w_1)$$
$$= 2 + 1 \times (2 - 1^2 + 1) = 4$$

Error Bounds

Theorem: If f is Lipschitz continuous on $D = [a,b] \times \mathbb{R}$ with const. L and if

$$|y''(t)| \le M \quad \forall t \in [a,b]$$

where y is the unique sol'n to

$\begin{cases} y' = f(t,y) & t \in [a,b] \\ y(a) = \alpha \end{cases}$

then $\quad |y(t_i) - w_i| \le \dfrac{hM}{2L}\left[e^{L(t_i - a)} - 1\right] \quad i = 0, \ldots, N$

Higher order Taylor methods:

Local Truncation error:

In Euler's method, we computed

$$w_{i+1} = w_i + h f(t_i, w_i)$$

More generally, we may use an iteration of the form

$$w_{i+1} = w_i + h\phi(t_i, w_i)$$

for some $f \neq \phi$ we choose.

The local truncation error is defined as

$$\tau_{i+1}(h) := \frac{\overbrace{y_{i+1}}^{y(t_{i+1})\text{ true value at } t_{i+1}} - \overbrace{(y_i + h\phi(t_i, w_i))}^{\text{predicted value at } t_{i+1} \text{ starting from } y_i}}{h}$$

$$= \frac{y_{i+1} - y_i}{h} - \phi(t_i, w_i)$$

Example: Euler's method has

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - \underbrace{f(t_i, y_i)}_{y'(t_i)}$$

$$= \frac{h}{2} y''(\xi_i) \quad \hookrightarrow \in [t_i, t_{i+1}]$$

$$= O(h)$$

~x~

This suggests improving upon Euler's method by using higher order Taylor approximations:

$$(*) \quad y(t_{i+1}) = y(t_i) + \sum_{i=1}^{n} \frac{h^i}{i!} y^{(i)}(t_i) + \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\xi_i)$$

$$(**) \quad y'(t) = f(t, y(t))$$

$$y''(t) = f'(t, y(t))$$

$$\vdots$$

$$y^{(k)}(t) = f^{(k-1)}(t, y(t))$$

(*) & (**) $\Longrightarrow$

$$y(t_{i+1}) = y(t_i) + \sum_{i=1}^{n} \frac{h^i}{i!} f^{(i-1)}(t_i, y(t_i))$$

$$+ \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))$$

If we ignore the error term, we have

Taylor methods of order n:

$$\boxed{\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + h \underbrace{T^{(n)}(t_i, w_i)} \end{cases}}$$

$f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, w_i)$

(Euler = Taylor of order 1)

Example: Derive Taylor's method of order 2 with $h=1$ to the IVP

$$\begin{cases} y' = y - t^2 + 1 & t \in [0,2] \\ y(0) = 0.5 \end{cases}$$

Sol'n: we have $f(t,y) = y - t^2 + 1$ and we need

$$f'(t,y) = \frac{d}{dt}\left(y(t) - t^2 + 1\right)$$

$$= \underbrace{y'(t)}_{f(t,y)} - 2t$$

$$= y - t^2 + 1 - 2t$$

So $T^{(2)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i)$

$$= w_i - t_i^2 + 1 + \frac{h}{2}\left(w_i - t_i^2 - 2t_i + 1\right)$$

$$= \left(1 + \frac{h}{2}\right)\left(w_i - t_i^2 + 1\right) - h t_i$$

$$\Rightarrow \begin{cases} w_0 = 0.5 \\ w_{i+1} = w_i + h\left[\left(1+\frac{h}{2}\right)(w_i - t_i^2 + 1) - h t_i\right] \end{cases}$$

Theorem: Taylor's method of order n has local truncation error $O(h^n)$ provided the sol'n $y \in C^n[a,b]$.

Advantages: • high order local truncation error

Disadvantage: Must compute & evaluate derivatives of $f(t,y)$